Non-linear residually finite groups

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Abstract

We give the first example of a non-linear residually finite 1-related group: $\langle a, t \mid a^{t^2} = a^2 \rangle$.

1 Non-linear groups

Let ϕ be an injective endomorphism of a group G. Then the HNN extension

$$\mathrm{HNN}_\phi(G) = \langle G, t \mid tgt^{-1} = \phi(g), g \in G \rangle$$

is called an *ascending HNN extension* of G (or the *mapping torus of the endomorphism* ϕ). In particular, the ascending HNN extensions of free groups of finite rank are simply the groups given by presentations $\langle x_1, ..., x_n, t \mid tx_it^{-1} = w_i, i = 1, ..., n \rangle$, where $w_1, ..., w_n$ are words in $x_1, ..., x_n$ generating a free subgroup of rank n.

In [BS], Borisov and Sapir proved that all ascending HNN extensions of linear groups are residually finite. After [BS], the question of linearity of these groups became very interesting. This question is especially interesting for ascending HNN extensions of free groups because most of these groups are hyperbolic [I.Kap], and because calculations show that at least 99.6% of all 1-related groups are ascending HNN extensions of free groups [BS].

Let $H = \mathrm{HNN}_\phi(F_n)$ be an ascending HNN extension of a free group. If $n = 1$ then H is a Baumslag-Solitar group $BS(m, 1)$, so it is inside $\mathrm{SL}_2(\mathbb{Q})$. If $n = 2$ and ϕ is an automorphism then the linearity of H follows from the linearity of $\mathrm{Aut}(F_2)$. The linearity of $\mathrm{Aut}(F_2)$ follows from two facts: Dyer, Formanek and Grossman [DFG] reduced the linearity of $\mathrm{Aut}(F_2)$ to the linearity of the braid group B_4; the linearity of B_4 was proved by Krammer [Kra].

It is known that in the case when ϕ is not an automorphism, the situation is different.

Proposition 1 (Wehrfritz, [W], Corollary 2.4). *The group* $\langle a, b, t \mid tat^{-1} = a^k, tbt^{-1} = b^l \rangle$, *with* $k, l \notin \{1, -1\}$, *is not linear.*

The proof[1] in [W] uses the action of $\mathrm{SL}_n(K)$ on the Lie algebra $\mathrm{sl}_n(K)$ to deduce that if a, b, t are matrices satisfying the relations of the group then some powers of a and b generate a nilpotent subgroup. One can also prove this statement by using the fact that if a, t are matrices with complex entries such that $tat^{-1} = a^k$, $|k| > 1$, then $a^{k-1} \in U(t^{-1}) = \{x \mid \lim_{n\to\infty} t^{-n}xt^n = \mathbf{1}\}$ and the well known fact that $U(t^{-1})$ is a nilpotent group for every matrix t.

The following lemma is useful when dealing with ascending HNN extensions of groups.

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[1]In the first version of this paper that appeared in the arXiv, we gave a complete proof of this statement because we were unaware of [W]. We are grateful to Professor Raptis for providing this reference.

Lemma 2. *Let ϕ be an injective endomorphism of a group G. Suppose that ϕ^k is not an inner automorphism of G for any $k \neq 0$. Then a homomorphism γ of $H = \mathrm{HNN}_\phi(G)$ is injective if and only if the restriction of γ on G is injective.*

Proof. This follows immediately from the fact [BS] that every element of H is of the form $t^{-p}wt^q$, where $p, q \geq 0$, $w \in G$. Indeed suppose that γ is injective on G but $\gamma(t^{-p}wt^q) = 1$ where $t^{-p}wt^q \neq 1$. Then $\gamma(wt^{q-p}) = 1$, and so $q \neq p$. Hence $\gamma(w) = \gamma(t^{p-q})$. Let $m = p - q$. We can assume that $m > 0$, otherwise replace w by w^{-1}. This implies that for every $u \in G$, $\gamma(t^m u t^{-m}) = \gamma(wuw^{-1})$. The injectivity of the restriction of γ on G then implies that $t^m u t^{-m} = wuw^{-1}$. Since $t^m u t^{-m} = \phi^m(u)$, we get $\phi^m(u) = wuw^{-1}$. Hence ϕ^m is the inner automorphism induced by w, a contradiction. $\square$

Corollary 3. *The group $H = \langle a, b, t \mid tat^{-1} = a^k, tbt^{-1} = b^l\rangle$ is linear if and only if $k, l \in \{1, -1\}$.*

Proof. If $k, l \in \{1, -1\}$ then ϕ is an automorphism and H is linear, say, by the results of [DFG] and [Kra] cited above (one can also use the fact that the group has a subgroup of finite index isomorphic to $F_2 \times \mathbb{Z}$). If both $k, l \notin \{1, -1\}$, we can apply Proposition 1. It remains to consider the case $k \notin \{1, -1\}$, $l \in \{1, -1\}$. Then it is easy to see by Lemma 2 that the subgroup $\langle a, bab^{-1}, t^2\rangle$ is isomorphic to the group $\langle x, y, t \mid txt^{-1} = x^{k^2}, tyt^{-1} = y^{k^2}\rangle$, and so it is not linear by Proposition 1. Hence the group H is not linear as well. $\square$

Not much is known about the linearity of 1-related groups. Note only that all residually finite Baumslag-Solitar groups (i.e. HNN extensions of cyclic groups) [Me] are linear [Vo].

The following theorem gives the first example of a non-linear residually finite 1-related group.

Theorem 4. *The group $\langle a, t \mid t^2at^{-2} = a^2\rangle$ is residually finite but not linear.*

Proof. Using Magnus rewriting procedure, this group can be represented as an HNN extension $\langle a, b, t \mid tat^{-1} = b, tbt^{-1} = a^2\rangle$, so it is residually finite by [BS]. The subgroup of that group generated by $\{a, b, t^2\}$ is isomorphic (by Lemma 2) to $\langle a, b, t \mid tat^{-1} = a^2, tbt^{-1} = b^2\rangle$ which is not linear by Proposition 1. The isomorphism takes a to a, b to b, t to t^2. $\square$

Problem 5. Is it true that $\mathrm{HNN}_\phi(F_n)$ is always linear if ϕ is an automorphism?

Problem 6. Are there hyperbolic non-linear ascending HNN extensions of free groups? In particular, is the group $\langle a, b, t \mid tat^{-1} = ab, tbt^{-1} = ba\rangle$ linear (the fact that this group is hyperbolic follows from [I.Kap])?

The group $\langle a, b, t \mid tat^{-1} = ab, tbt^{-1} = ba\rangle$ is actually a 1-related group $\langle a, t \mid [[a, t], t] = a\rangle$. The fact that this group does not have a faithful 2-dimensional representation follows from [FLR]. Moreover, results of [FLR] (and prior results of Magnus [M]) imply that most 1-related groups do not have faithful 2-dimensional representations.

We conjecture that the answer to Problem 6 is that most groups $HNN_\phi(F_k)$, $k \geq 2$, are non-linear provided ϕ is not an automorphism. In particular there are many non-linear hyperbolic groups among these HNN extensions.

Note that there are examples of non-linear hyperbolic groups yet, although M.Kapovich [M.Kap] has an example of a hyperbolic group which does not have faithful real linear representations.[2]

[2]Added 10/12/04: It is easy to show using Kapovich's argument that the group does not have faithful representations over any field.

Proposition 1 and Corollary 3 give examples of non-linear ascending HNN extensions of linear groups. Non-ascending HNN extensions with this property are much easier to find: some of these HNN extensions are not even residually finite (say, the Baumslag-Solitar groups $\langle a, t \mid ta^2t^{-1} = a^3 \rangle$). Residually finite non-linear HNN extensions of linear groups were constructed in particular by Formanek and Procesi [FP]. They proved that the HNN extension of the direct product $F_k \times F_k$, where one of the associated subgroups is the diagonal and the other one is one of the factors, is residually finite but not linear. This was the main ingredient in the proof in [FP] of the non-linearity of $\mathrm{Aut}(F_n)$, $n \geq 3$.

2 Representations in $\mathrm{SL}_2(\mathbb{C})$

By Lemma 2, finding a copy of $H = \mathrm{HNN}_\phi(F_k)$ in $SL_n(K)$ amounts to finding a k-tuple of matrices $(A_1, ..., A_k)$ that freely generate a free subgroup, and which is a conjugate of the k-tuple $(\phi(A_1), ..., \phi(A_k))$. In the case when $k = 2$, $n = 2$, $K = \mathbb{C}$ one can use the fact that conjugacy of two pairs of 2 by 2 matrices (A, B), (C, D) implies the system of equalities $\mathrm{trace}(A) = \mathrm{trace}(C), \mathrm{trace}(B) = \mathrm{trace}(D), \mathrm{trace}(AB) = \mathrm{trace}(CD)$. The converse statement is "almost" true: one needs to exclude the case when

$$\mathrm{trace}([A,B]) = \mathrm{trace}(A)^2 + \mathrm{trace}(B)^2 + \mathrm{trace}(AB)^2 - \mathrm{trace}(A)\mathrm{trace}(B)\mathrm{trace}(AB) - 2 = 2 \quad (1)$$

(in that case A, B generate a solvable group [Bow]). Using the fact that for every word $u = u(A, B)$ in matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$, $\mathrm{trace}(u)$ can be expressed as a polynomial in $\mathrm{trace}(A)$, $\mathrm{trace}(B)$, $\mathrm{trace}(AB)$, we get a system of three equations with three unknowns. The corresponding algebraic variety will be called the *trace variety* of the group $\mathrm{HNN}_\phi(F_k)$.

In most cases that we considered, the trace variety was 0-dimensional. But the next example shows that the trace variety may have dimension ≥ 1 and the group still may not have a faithful 2-dimensional representation.

Example 7. *The group $H = \langle a, b, t | tat^{-1} = a, tbt^{-1} = [a, b] \rangle$ does not have a faithful 2-dimensional representation. The trace variety of this group is a union of two curves, but it consists of non-faithful representations.*

Consider any representation of H in $\mathrm{SL}_2(\mathbb{C})$. So we assume that a, b are 2 by 2 matrices with determinant 1. Let us denote $\mathrm{trace}(a) = x, \mathrm{trace}(b) = y, \mathrm{trace}(ab) = z$. It is easy to see using

$$\mathrm{trace}(BA^2C) = \mathrm{trace}(A)\mathrm{trace}(BAC) - \mathrm{trace}(BC) \quad (2)$$

(this is essentially the Cayley-Hamilton theorem for matrices in SL_2) that we have the following system of equations:

$$\begin{cases} x = x \\ y = \mathrm{trace}(aba^{-1}b^{-1}) = -2 + x^2 + y^2 + z^2 - xyz \\ z = \mathrm{trace}(a[a,b]) =_{\mathrm{by}(2)} x \cdot \mathrm{trace}(aba^{-1}b^{-1}) - \mathrm{trace}(ba^{-1}b^{-1}) \\ \quad = x \cdot \mathrm{trace}(aba^{-1}b^{-1}) - x = xy - x \end{cases}$$

Plugging $z = xy - x$ into the second equation, and solving for y, we get $y = 2, x = z$ or $y = x^2 - 1, z = x^3 - 2x$. Thus the trace variety is a union of two curves. If $y = 2, x = z$ then $\langle a, b \rangle$ is solvable by (1), so the representation is not faithful. Now let $y = x^2 - 1, z = x^3 - 2x$. Consider the word $w = ab^{-1}a^{-1}ba^{-1}b^{-1}a$. It is not difficult to compute the corresponding trace polynomial:

$$\mathrm{trace}(w) = -3y - 4xz + 5yx^2 + xz^3 - 2yx^2z^2 + yz^2 + y^3 - y^3x^2 + y^2x^3z + x^3z - yx^4.$$

If we plug in $y = x^2 - 1, z = x^3 - 2x$ into this polynomial, we get 2. Similarly, the trace polynomial of the word wa is

$$x^4y^2z - x^5y - x^3y^3 - 2x^3yz^2 + x^4z - x^2y^2z + x^2z^3 + 6x^3y + 2xy^3 + 3xyz^2 - 5x^2z - y^2z - z^3 - 7xy + 3z.$$

If we plug in $y = x^2 - 1, z = x^3 - 2x$, we get x. Hence $\mathrm{trace}(w) = 2, \mathrm{trace}(wa) = \mathrm{trace}(a) = x$, whence $\mathrm{trace}([w, a]) = 2$ (see (1)) and w and a generate a solvable subgroup. Therefore for every value of x the corresponding representation of the group H is not faithful. (In fact it is not difficult to show that the relation $(a^2b)^3 = 1$ also holds, so in this case $\langle a, b\rangle$ has torsion.)

Similarly the trace variety of the group $\langle a, b, t \mid tat^{-1} = a, tbt^{-1} = (ba)b(ba)^{-1}\rangle$ is two-dimensional, but this group does not have faithful representations in $\mathrm{SL}_2(\mathbb{C})$.

In fact we do not know the answer to the following question.

Problem 8. Are there any ascending HNN extensions of F_k, $k > 1$, which have faithful 2-dimensional complex representations? In particular, are there free non-cyclic subgroups in $\mathrm{SL}_2(\mathbb{C})$ which are conjugate inside $\mathrm{SL}_2(\mathbb{C})$ to their proper subgroups?

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